Long Term Plan: Further Maths

Learning Cycle Year 1	Knowledge and Skills	Vocabulary & Reading	Checking of understanding	Rationale
Autumn Term	Pure – Matrices • Matrix addition, subtraction and multiplication • Inverse of 2 × 2 and 3 × 3 matrices • Simultaneous Equations • Linear transformations Pure – Complex Numbers 1 • Introduction of complex numbers, basic manipulation • Complex conjugate, division and solving polynomial equations • Argand diagrams • Modulus and argument Pure - Roots of polynomials 1 • Roots of a quadratic and cubic • Formation of polynomial equations • Linear Transformations of roots Decision – Graph theory • Modelling with graphs • Basic Terminology and special graphs • Representing graphs using matrices Statistics – Discrete random variables • Expected value and variance of X	Array; Elements; Size Square Matrix; Zero Matrix; Leading Diagonal; Identity Matrix; Order Singular/Non-singular matrix; Determinant; Transpose; Matrix of Minors Matrix of Cofactors; Rule of alternating signs; Self-inverse; Consistent/Inconsistent; Sheaf; Prism; Linear multiple Imaginary number Complex number Real and Imaginary part Coefficient Complex conjugation Argand diagram Modulus; Argument Principal argument Coefficient; Root; Quadratic and Cubic Equation; Quartic	Throughout the year there will be • Weekly homework from both teachers – by topic and mixed questions including problem-solving. Self-assessment using purple pen. • Classroom strategies will include: Question and Answer, both open questioning and closed questioning, differentiated as appropriate • Use of mini-whiteboards to aid Q and A • Teacher scrutiny of work and exercises, performed in real time. • Folder scrutiny	Matrices will occur at this initial point in time, since it does not require a great deal of background knowledge from Year 12 Mathematics, unlike much of the Year 12 further Mathematics curriculum (notably calculus and trigonometry). The same is true of Complex Numbers, which can largely be taught with reference to GCSE knowledge. Roots of polynomials is a topic which similarly does not require a great deal of background knowledge beyond what is learned at GCSE. However Quadratic Equations is taught at the outset of year 12 in mathematics, which provides a good foundation for this work

	and a function of X • Solving problems. Decision – Algorithms on Graphs • Kruskal's and Prim's algorithm • Using Dijkstra's algorithm to find the shortest path • Eulerian graphs and Route Inspection algorithm Pure – Series • Sum of natural numbers • Sum of squares and cubes Statistics – Poisson Distribution 1 • Modelling and mean and variance • Adding Poisson distributions	Equation; Reciprocal; Linear Transformation; Graph; Network; Vertex Node; Edge; Arc; Weight Subgraph; Degree; Valency Walk; Path; Trail; Cycle; Hamiltonian cycle; Eulerian circuit; Connected graph; Directed graph; Digraph; Tree; Spanning tree; Complete graph; Isomorphic graph; Adjacency matrix; Distance matrix Discrete Random Variable; Expectation; Variance Minimum spanning tree; Minimum connector; Algorithm; Prim's and Kruskal's algorithms; Dijkstra's algorithm Sigma notation; Natural numbers Poisson distribution "e"; The exponential function; Statistical Independence; "Singly"	Concept and skill check opportunities (Integral Maths): Core Pure M1-3 & I1-3 – Matrices C1-2 – Complex Numbers R1-2 – Roots of Polynomials Decision A1-2 - Algorithms N1-3 – Networks Statistics D1-2 – Discrete random variables Summative tests TEST 1 to include: • Matrices • Complex Numbers • Roots of polynomials TEST 2 to include: • Discrete random variables • Graphs and Algorithms • Linear transformations • Algorithms • Series • Recall from T1	Decision Mathematics is included in the curriculum because we consider this area of Mathematics to have many applications in the "real world". Many careers require mathematical skills; often this should be interpreted as understanding Decision Maths skills and not simply numeracy. Graph Theory is an excellent place to start, since it provides the contrast with algebraic mathematics, with which the students are most familiar. Discrete Random Variables is an accessible place to start the Statistics component of the Further Maths course. It provides the foundation for the Poisson Distribution and distributions studied in Year 13. GCSE skills are all that is required to proceed. Algorithms on Graphs are a concrete introduction to the concept of an algorithm in Decision

				Maths. These will also be a fun and novel way of performing maths for many students. Some knowledge of the previous Decision Mathematics topic – Graph Theory - is required. Series is a challenging algebraic topic, which from this point of view is well positioned at this time; the students will have gained some confidence to engage with this topic, as well as being challenged algebraically by it. The Poisson distribution is separated into two sections to encourage recall. The first half of the topic is taught at this time to offer some contrast to the Pure Mathematics and Decision Maths that has been covered to date and as a reminder that Statistics is an important component of the course.
Spring Term	Pure – Vectors • Vector and Cartesian equations of a line and a plane • Scalar product	Scalar parameter; Position vector; direction vector; Cartesian vs Vector form of a line/plane; Collinear;	Concept and skill check opportunities (Integral Maths): Core Pure	The Vectors section covers a lot of material and so

	• Scalar product form of the equation of a plane • Points of Intersection • Perpendicular vector and distance • Problems involving points, lines and planes Pure – Proof by Induction • Proving results about sum of a series, divisibility and matrices Pure – Complex Numbers 2 • Loci and regions in the Argand diagram Decision – Linear Programming • Formulating a problem • Graphical methods • Locating the optimal point and solutions with integer values Decision – Critical path analysis • Modelling a project • Dummy activities • Early and late event times • Critical activities • Gantt charts Statistics – Poisson Distribution 2 • Mean and Variance of binomial • Poisson as an approximation to binomial • Hypothesis Testing	Normal vector; Scalar (dot) product; Skew lines; Perpendicular distance. Basis, Assumption, Inductive and Conclusion steps Modulus-Argument form; Principal argument; Locus/Loci; Half-line Decision variable; Objective function; Constraints; Feasible solution; Feasible region; Optimal solution; Vertex testing Activity network; Precedence table; Dependence table; Source node; Sink node; Dummy activity; Forward/Backward pass/scan; Critical path; Float; Gantt chart Null hypothesis; Alternative hypothesis; Significance level; One or Two tailed; Critical region/ value	S1-2 – Sequences and series VR1 – Volumes of Revolution V1-2 – Vectors G1-2 – Loci Decision L1 – Linear Programming N1-3 – Networks C1-2 – Critical Path Analysis Statistics DD1-2 – Discrete distributions Summative tests TEST 3 to include: • Poisson Distribution • Proof by Induction • Vectors • Complex Numbers • Recall from T1 & T2 TEST 4 – AS Decision Mock	takes some time. The students have been exposed to Vectors in the Maths course. Proof by Induction is a form of proof often seen in mathematics degrees. Aspects of this can be challenging, and require the students to be algebraically competent. Hence its inclusion at this stage of the academic year. A revisit of Complex Numbers to enable recall of work covered at the beginning of year 12, but the material is sophisticated and is appropriately included at this time when the students have had the opportunity to develop mathematical skills and benefit from prior knowledge; e.g. the Binomial Expansion (Maths). The Decision Mathematics topics – Linear Programming and Critical Path Analysis require some previous knowledge [Inequalities (Maths) and

	Pure – Volume of Revolution • around the x-axis • around the y-axis • modelling Statistics – Chi-squared tests • Testing the goodness of fit with discrete data • Using contingency tables Pure – Roots of Polynomials 2 • Roots of a quartic.	Radian measure Goodness of fit; Observed and Expected frequency; Degrees of freedom; Constraint/Restriction; Uniform distribution; Contingency tables		Networks (D1) respectively]. These conclude the Decision Mathematics aspects of the Year 12 course. Volume of Revolution can be addressed at this stage now, since Integration, which (knowledge of) is required has been covered in the mathematics course. Chi-Squared is the final aspect of Statistics which needs to be covered, This requires knowledge of hypothesis testing, which has been seen previously (Poisson Distribution 2). Roots of Polynomials (Quartics) has been included at the end of the AS course to enable recall, but also offers depth to knowledge of this topic introduced in Term 1.
Summer Term	Pure – Series • Method of differences Pure – Methods of Calculus • Rules of differentiation; chain rule, product rule, quotient rule •differentiating trigonometric functions, exponentials and logarithms.	Partial fractions; Method of differences Chain rule; Product rule; Quotient rule; Derivative	AS FURTHER MATHS EXAM	The AS Exam takes place at this point. All Year 12 material has been covered. Series – Method of Differences is an accessible Year 13 topic. It does

	Decision – Algorithms on Graphs • Planarity algorithm • Floyd's algorithm Pure – Complex Numbers • Exponential form • Multiplying and dividing • De Moivre's theorem	Planar vs Non-planar graph; Hamiltonian cycle; Distance table; Route table De Moivre's Theorem; Exponential form of a complex number; n-gon; nth roots of unity		require easy Partial Fractions to be taught (a year13 topic). Calculus is central to the year 13 FM course, so differentiation techniques are pre-requisite knowledge, hence the inclusion of differentiation here. For Decision Maths, the Planarity and Floyd's Algorithms are accessible start points for year 13. The Complex Numbers addressed here requires year 12 knowledge and is a good introductory topic for year 13.

Learning Cycle Year 2	Knowledge and Skills	Vocabulary & Reading	Checking of understanding	Rationale
Autumn Term	Pure - Hyperbolic Functions • Hyperbolic and Inverse hyperbolic functions • Identities and equations • Differentiation and Integration Pure – Complex Numbers • Know and use $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ • De Moivre's theorem • The nth roots of $z = re^{i\theta}$ and complex roots of unity Pure – Series continued • Higher derivatives • Maclaurin Series • Expansions of compound functions Pure – Integration • Techniques of integration (Year 2 Pure) Statistics – Geometric and Negative Binomial distributions • Properties and application • Mean and variance of each Decision – The travelling salesman problem	Sinhx, coshx, tanhx, sechx, cosechx, cothx Arsinhx, arcoshx, artanhx etc. Osborn's rule De Moivre's Theorem; Modulus-Argument form; Euler's relation; Euler's Identity Higher derivatives; n^{th} derivative; d^n/dx^n vs. d^ny/dx^n Maclaurin series, Alternating sequences; Cf. Year 13 Maths LTP Geometric distribution; Negative binomial distribution; Mean and Variance	Throughout the year there will be • Weekly homework from both teachers – by topic and mixed questions including problem-solving. Self-assessment using purple pen. • Classroom strategies will include: Question and Answer, both open questioning and closed questioning, differentiated as appropriate • Use of mini-whiteboards to aid Q and A • Teacher scrutiny of work and exercises, performed in real time. • Folder scrutiny	Hyperbolic Functions can be introduced at the beginning of the year as the students will be familiar with the basic concepts (exponentials) from the end of year 12 maths. Identities and equations are also possible to consider at this stage. The calculus will be considered later, when the students have sufficient differentiation and Integration skills (to be taught this term). The Complex Numbers here follows on directly from the material learned in year 12. Inclusion at this stage will enable recall, but offers an appropriate level of challenge at the beginning of the year. Calculus is not required knowledge.

	• Classical and practical problem • Using a minimum spanning tree; method to find an upper and lower bound • Nearest neighbour algorithm Statistics – Tests for Geometric Distributions • Hypothesis tests for parameter p • Goodness of fit test Decision – Linear Programming (Review) Statistics – Probability Generating Functions • Use of Probability Generating Functions for Negative Binomial, Geometric, Binomial and Poisson distributions • Probability generating function of the sum of independent random variables Pure – Volume of Revolution • Around the x-axis and y-axis • Using parametrically defined curves • Modelling	Classical or Practical problem; Upper and Lower bounds; Triangle inequality; Shortcut; Residual spanning tree; Nearest neighbour algorithm; Null and Alternative hypotheses; One and Two-tailed tests Cf. Year 12 Further Maths Dummy variable; PGF; Independent random variables (No new terminology)	Concept and skill check opportunities (Integral Maths): Core Pure S1 – Method of differences C1-2 – Complex Numbers H1-2 – Hyperbolic Functions Decision FG1-2 – Further algorithms on graphs Statistics FD1-2 – Further discrete distributions H1 – Hypothesis testing C1-2 – Chi-squared tests G1 – Probability Generating functions Summative tests TEST 1 to Include: • Hyperbolic Functions • Differentiation • Maclaurin Series • Complex Numbers TEST 2 to include: • Recall from T1	For the Maclaurin Series, the required differentiation techniques have been learned at the end of year 12. This is a relatively accessible topic so appropriate for the first few weeks of the term. Integration needs to be taught at this stage, since it is required knowledge for most of the remainder of the pure sections of the FM course. (See Year 13 Maths LTP for details) This is an appropriate time to reintroduce Statistics into the course. Subsequent statistics topics require knowledge of these two distributions. The Travelling Salesman problem is a sophisticated problem in Decision Maths. This is a continuation of the work that has been done to date on networks and network algorithms. Some further Statistics continues the interleaving of Statistics and Decision Maths with the Pure topics

				to be taught. The students will now be familiar with the language and protocols of hypothesis testing. The inclusion of a Linear Programming review is as a prelude to the Simplex Algorithm (term 2). This also allows for recall of year 12 material. Probability Generating Functions encourages an in depth understanding of the mathematics behind the various distributions that are encountered in A-level maths and A-level further maths Volume of Revolution problems are often given in context and provide a good opportunity for problem solving. The required integration results and techniques are now known by the students. Parametrically defined curves will be considered later on, when this has been taught in year 13 mathematics.

Spring Term	Decision – Simplex Algorithm • Method and problems involving integer solutions • Two-stage simplex method • The Big-M method Pure – Methods in Calculus • Improper integrals • Mean value of a function • Integrate using partial fractions • Differentiate and Integrate inverse trigonometric functions and integrate using trigonometric substitutions Pure – Polar Coordinates • Convert between Cartesian and polar and sketch r(θ) • Area enclosed by a polar curve • Tangents to polar curves Pure – Methods in Differential Equations • Integrating factors to solve first order differential equations • Second order differential equations • Boundary conditions Statistics – Central Limit Theorem • Central Limit Theorem • Application to other distributions	Simplex method; Slack variables; Pivot row; Simplex tableaux; Basic feasible solution; Basic variables, Surplus variables, Non-basic variables; Artificial variables; Two-stage method; Big-M method Improper integral; Convergent; Divergent; Infinite limit; Mean value (of a function) Pole; Initial line; Half-line; Spiral; Polar curve; Area of a sector First-order differential equation; Separating the variables; Integrating factor; Second-order differential equation; homogeneous vs non-homogeneous; Auxilliary equation; Complementary function; Particular Integral; General and Particular solutions; Boundary conditions Central limit theorem; Sample mean; Distribution of the sample mean; Population vs Sample	Concept and skill check opportunities (Integral Maths): Pure P1-2 – Polar Coordinates F1-2 – First Order Differential Equations SD1-3 – Second Order Differential Equations FC1-3 – Further Calculus Decision FG3 – Critical Path Analysis Statistics L1 – Central Limit Theorem Summative tests TEST 3 - Mock Exam All content taught to date.	A more sophisticated version of Linear Programming. It is possible to highlight real world problems that can be solved by applying the Simplex algorithm. This is at times better taught in a "flipped" way, as each algorithm is intricate and time-consuming. Methods in Calculus continues to build upon and develop calculus knowledge and skills. Polar coordinates is a topic area, which is conceptually arresting and novel for the students. It poses an intellectual challenge therefore, and is best addressed later on in the course. There is also a lot of pre-requisite knowledge needed, so this is quite a synoptic area of study. Differential Equations do require very good integration ability; there are also a lot of distinct aspects to this topic which need to

	Pure – Modelling with Differential Equations • First order differential equations • Simple harmonic motion • Damped and forced harmonic motion • Coupled first-order differential equations Statistics – Quality of Tests • Type I and Type II errors • Size and power of test Decision - Critical Path Analysis (continued from year 12) • Review year 12 • Resource Histograms • Scheduling Pure – Volume of Revolution • Using parametrically defined curves	Simple harmonic motion; Angular velocity; Centre of oscillation; Period; Amplitude; Damped harmonic motion; Heavy damping; Critical damping; Light damping; Damping force; Forced harmonic motion; Coupled first-order differential equations Type I error; Type II error; Actual significance level; Sample variance; Size of a test; Power of a test; Power function of a test Resource histogram; Levelling; Scheduling; Gantt Chart (No new terminology)		be remembered. Again, a synoptic topic suited to the end of the course. For the Central Limit Theorem, a very good knowledge of all of the statistical distributions studied to date in maths and further maths is needed: Binomial; Poisson; Normal; Negative Binomial and Geometric distributions. It serves as a good revision and recall enabling topic therefore, and is synoptic in nature. Modelling with Differential Equations has several virtues: real world contexts in biology, chemistry and engineering and problem-solving opportunities; synoptic in nature and the topic links closely to Physics, (often FM students also study physics). This area of the course follows directly on from differential equations (above). For a critical understanding of Statistics an appreciation of the quality of statistical tests is

				important. From a career point of view, the ability to form a critical appraisal of statistics is invaluable. A review and conclusion of the Critical Path Analysis started in year 12. Its inclusion at this stage allows for a review and recall of a challenging topic prior to the end of the course. The conclusion of Volumes of Revolution. Since parametric integration has been taught in A-level maths by now, this aspect of the course can be addressed.
Summer Term	REVISION in preparation for the A-Level exams			